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Negative semi-discrete KP and BKP hierarchies via nonlocal symmetries

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Abstract
In this paper, nonlocal symmetries for the bilinear semi-discrete Kadomtsev–Petviashvili (KP) and bilinear semi-discrete B-type KP (BKP) equations are derived. By expanding these nonlocal symmetries in power series of each of two parameters, we have derived bilinear negative semi-discrete KP and BKP hierarchies. An interesting observation is that bilinear positive and negative semi-discrete KP hierarchies may be derived from the same nonlocal symmetry for the semi-discrete KP equation.

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1. Introduction

In the literature, a specific but important class of nonlocal symmetries called eigenfunction symmetries and their applications have intensely been studied (see, e.g. [1–8]). In [9], Lou has derived a hierarchy of negative Kadomtsev–Petviashvili (KP) equations from eigenfunction symmetry of the KP equation. Very recently, nonlocal symmetries for the bilinear KP and B-type KP (BKP) equations have been revisited. By expanding these nonlocal symmetries, we have derived two types of bilinear negative KP hierarchies and two types of bilinear negative BKP hierarchies [10]. An interesting thing is that bilinear positive and negative KP (and BKP) hierarchies may be derived from the same nonlocal symmetries for the KP (and BKP) equations. Based on these observations, it is quite natural for one to study nonlocal symmetries for the semi-discrete KP and semi-discrete BKP equations and their applications.

The semi-discrete KP equation,

$$\Delta(u_{n,t} + 2u_{n,y} - 2u_n u_{n,y}) - (2 + \Delta)u_{n,yy} = 0, \tag{1}$$

is first proposed by Date *et al* [11], where u is a function depending on one discrete variable n and two continuous ones y and t, and Δ denotes the forward difference operator defined by $\Delta f_n \equiv f_{n+1} - f_n$. Concerning the semi-discrete KP equation (1), various properties have already been established, such as infinitely many symmetries (conservation laws) [12], *N*-soliton solutions [13], singularity confinement [14], matrix integral solution [15] and so on. Recently, two generalizations of equation (1), the pfaffianized semi-discrete KP equation and the semi-discrete KP equation with self-consistent sources, are proposed respectively [16, 17], whose integrability is confirmed by the existence of *N*-soliton solutions and Bäcklund transformations.

To our knowledge, relatively less attention has been paid to the semi-discrete BKP equation

$$u_{n+1,y} - u_{n,y} = u_{n+1,xx} + u_{n,xx} + (u_{n+1,x} + u_{n,x} + 1)(u_{n+1,x} - u_{n,x}) \\ - \tfrac{1}{2}(\mathrm{e}^{u_{n+2}-u_n} - \mathrm{e}^{u_{n+1}-u_{n-1}}), \tag{2}$$

which first appeared in [11]. In [18], the Pfaffian solution of equation (2) was given. Besides, the semi-discrete BKP equation with self-consistent sources has been proposed in the same paper. It is shown that the semi-discrete BKP equation with self-consistent sources is integrable in the sense of the existence of *N*-soliton solutions and Bäcklund transformation.

The purpose of this paper is first to derive nonlocal symmetries for the bilinear semi-discrete KP and bilinear semi-discrete BKP equations and then to apply these nonlocal symmetries to generate negative semi-discrete KP hierarchies and negative semi-discrete BKP hierarchies. The content of the paper is organized as follows. In section 2, we will consider nonlocal symmetries with three different parameters for the bilinear semi-discrete KP equation and then use these symmetries to generate negative and positive semi-discrete KP hierarchies in bilinear form. Section 3 is devoted to considering nonlocal symmetries with three different parameters for the bilinear semi-discrete BKP equation and then use these symmetries to produce negative and positive semi-discrete BKP hierarchies in bilinear form. Conclusions and discussions are given in section 4.

2. Nonlocal symmetries for the bilinear semi-discrete KP equation and its application

By the dependent variable transformation [13]

$$u_n = \frac{\partial}{\partial y} \ln \frac{f_{n+1}}{f_n} \tag{3}$$

the semi-discrete KP equation (1) is transformed to the bilinear form

$$\left(D_t + 2D_y - D_y^2\right) \mathrm{e}^{\frac{1}{2}D_n} f_n \cdot f_n = 0, \tag{4}$$

where D is the Hirota bilinear operator [19]

$$D_y^m D_t^n f \cdot g = \frac{\partial^m}{\partial y^m} \frac{\partial^n}{\partial t^n} f(y+\epsilon, t+\delta) g(y-\epsilon, t-\delta)|_{\epsilon=0,\delta=0}$$

$$\mathrm{e}^{\delta D_n} f_n \cdot g_n = f_{n+\delta} g_{n-\delta}.$$

Concerning (4), we have two sets of bilinear Bäcklund transformations given by

$$(D_y \mathrm{e}^{\frac{1}{2}D_n} + \lambda \mathrm{e}^{-\frac{1}{2}D_n} + \theta \mathrm{e}^{\frac{1}{2}D_n}) f_n \cdot g_n = 0 \tag{5}$$

$$\left(D_t + D_y^2 + 2(1+\theta) D_y + \gamma\right) f_n \cdot g_n = 0 \tag{6}$$

and

$$(D_y \mathrm{e}^{\frac{1}{2}D_n} + \lambda \mathrm{e}^{-\frac{1}{2}D_n} + \theta \mathrm{e}^{\frac{1}{2}D_n}) h_n \cdot f_n = 0 \tag{7}$$

$$\left(D_t + D_y^2 + 2(1+\theta) D_y + \gamma\right) h_n \cdot f_n = 0 \tag{8}$$

which are inferred from the bilinear Bäcklund transformation for the semi-discrete KP equation with self-consistent sources [17]. Here λ, θ and γ in (5)–(8) are arbitrary parameters. We now have the following result.

Proposition 1. *If g_n and h_n satisfy (5)–(8), then σ_n satisfying*

$$\Delta\left(\frac{\sigma_n}{f_n}\right) = \phi_n\psi_{n+1} \tag{9}$$

$$\left(\frac{\sigma_n}{f_n}\right)_y = \lambda\phi_n\psi_n \tag{10}$$

$$\left(\frac{\sigma_n}{f_n}\right)_t = \lambda^2(\phi_{n+1}\psi_n + \phi_n\psi_{n-1}) + \lambda(v_{n+1,y} - v_{n-1,y} - 2)\phi_n\psi_n \tag{11}$$

is a non-local symmetry of the bilinear semi-discrete KP equation (4), where

$$\phi_n = \frac{g_n}{f_n}, \qquad \psi_n = \frac{h_n}{f_n}, \qquad v_n = \ln f_n. \tag{12}$$

namely, σ_n satisfies the symmetry equation

$$\left(D_t + 2D_y - D_y^2\right) \mathrm{e}^{\frac{1}{2}D_n}(f_n\cdot\sigma_n + \sigma_n\cdot f_n) = 0. \tag{13}$$

Proof. Under transformations (12), the bilinear semi-discrete KP equation (4) is transformed to

$$\Delta v_{n,t} + 2\Delta v_{n,y} - (\Delta v_{n,y})^2 - (2+\Delta)v_{n,yy} = 0, \tag{14}$$

while the two sets of bilinear Bäcklund transformation (5)–(8) are transformed to, respectively,

$$\phi_{n,y} = \lambda\phi_{n+1} + \theta\phi_n + \phi_n\Delta v_{n,y} \tag{15}$$

$$\phi_{n,t} = \phi_{n,yy} + 2v_{n,yy}\phi_n - 2(1+\theta)\phi_{n,y} + \gamma\phi_n \tag{16}$$

and

$$\psi_{n+1,y} = -\theta\psi_{n+1} - \lambda\psi_n - \psi_{n+1}\Delta v_{n,y} \tag{17}$$

$$\psi_{n+1,t} = -\psi_{n+1,yy} - 2v_{n+1,yy}\psi_{n+1} - 2(1+\theta)\psi_{n+1,y} - \gamma\psi_{n+1}. \tag{18}$$

Utilizing the above equations, it is straightforward to check that (9)–(11) are compatible. Furthermore, let $w_n = \sigma_n/f_n$ for convenience, then we have

$$\begin{aligned}
&\frac{1}{f_{n+1}f_n}\left(D_t + 2D_y - D_y^2\right)\mathrm{e}^{\frac{1}{2}D_n}(f_n\cdot\sigma_n + \sigma_n\cdot f_n)\\
&\quad= \Delta w_{n,t} + 2\Delta w_{n,y} - 2(\Delta w_{n,y})(\Delta v_{n,y}) - (2+\Delta)w_{n,yy}\\
&\qquad+ [\Delta v_{n,t} + 2\Delta v_{n,y} - (\Delta v_{n,y})^2 - (2+\Delta)v_{n,yy}](w_{n+1} + w_n)\\
&\quad= \Delta w_{n,t} + 2\Delta w_{n,y} - 2(\Delta w_{n,y})(\Delta v_{n,y}) - (2+\Delta)w_{n,yy}.
\end{aligned}$$

Substituting (9)–(11) into the above expression, we conclude that the symmetry equation holds. □

For convenience, we formally denote σ_n as

$$\sigma_n = f_n\Delta^{-1}\frac{g_nh_{n+1}}{f_{n+1}f_n}. \tag{19}$$

In the following, we would like to present negative semi-discrete KP hierarchies.

Case 1.

$$\theta = 0, \qquad g_n = \sum_{i=0}^{\infty} g_n^{(i)}\lambda^i, \qquad h_n = \sum_{i=0}^{\infty} h_n^{(i)}\lambda^i$$

One negative semi-discrete KP hierarchy is given by

$$f_{n,t_{-m}} = f_n\Delta^{-1}\left[\frac{1}{f_{n+1}f_n}\sum_{i=0}^{m} g_n^{(i)}h_{n+1}^{(m-i)}\right]$$
$$(D_y\,\mathrm{e}^{\frac{1}{2}D_n}+\lambda\,\mathrm{e}^{-\frac{1}{2}D_n})f_n\cdot g_n = 0$$
$$(D_y\,\mathrm{e}^{\frac{1}{2}D_n}+\lambda\,\mathrm{e}^{-\frac{1}{2}D_n})h_n\cdot f_n = 0.$$

Then, we have

$$D_{t_{-m}}f_{n+1}\cdot f_n = \sum_{i=0}^{m} g_n^{(i)}h_{n+1}^{(m-i)}, \tag{20}$$

$$D_y\,\mathrm{e}^{\frac{1}{2}D_n}f_n\cdot g_n^{(i)} = -\mathrm{e}^{-\frac{1}{2}D_n}f_n\cdot g_n^{(i-1)}, \tag{21}$$

$$D_y\,\mathrm{e}^{\frac{1}{2}D_n}h_n^{(i)}\cdot f_n = -\mathrm{e}^{-\frac{1}{2}D_n}h_n^{(i-1)}\cdot f_n \tag{22}$$

with $g_n^{(-1)} = h_n^{(-1)} = 0$. By the dependent variable transformation (12), we may further transform (20)–(22) into the following nonlinear form:

$$\Delta v_{n,t_{-m}} = \sum_{i=0}^{m}\phi_n^{(i)}\psi_{n+1}^{(m-i)}, \tag{23}$$

$$\phi_{n,y}^{(i+1)} = \phi_{n+1}^{(i)} + \phi_n^{(i+1)}\Delta v_{n,y}, \tag{24}$$

$$\psi_{n+1,y}^{(i+1)} = -\psi_n^{(i)} - \psi_{n+1}^{(i+1)}\Delta v_{n,y} \tag{25}$$

with $\phi_n^{(-1)} = \psi_n^{(-1)} = 0$. From (24) and (25) we see that $\phi_n^{(i+1)}$ and $\psi_n^{(i+1)}$ can be solved by y-integration in terms of $\phi_{n+1}^{(i)}$ and $\psi_{n-1}^{(i)}$, respectively. That is why we call (23)–(25) one negative semi-discrete KP hierarchy.

In the following, through examples, we exhibit that equations (20)–(22) can be transformed to a set of bilinear equations with only one τ-function by introducing additional variables $z_1, z_2, \ldots$

Example 1. $m = 1$. In this case, we set $g_n^{(0)} = f_{n+1}, h_n^{(0)} = f_{n-1}, g_n^{(1)} = f_{n+1,z_1}$ and $h_n^{(1)} = -f_{n-1,z_1}$; then the t_{-1} -flow of the negative semi-discrete KP hierarchy becomes

$$D_{t_{-1}}\,\mathrm{e}^{\frac{1}{2}D_n}f_n\cdot f_n = D_{z_1}\,\mathrm{e}^{\frac{1}{2}D_n}f_n\cdot f_n \tag{26}$$

$$D_yD_{z_1}f_n\cdot f_n = 2\,\mathrm{e}^{D_n}f_n\cdot f_n \tag{27}$$

which means we may choose $z_1 \equiv t_{-1}$. In this case, (22) and (23) reduce to

$$D_yD_{t_{-1}}f_n\cdot f_n = 2\,\mathrm{e}^{D_n}f_n\cdot f_n \tag{28}$$

which is nothing but the bilinear form for the Toda molecule equation [19].

Example 2. $m = 2$. In this case, we set

$$g_n^{(0)} = f_{n+1},\qquad h_n^{(0)} = f_{n-1},\qquad g_n^{(1)} = f_{n+1,z_1},\qquad h_n^{(1)} = -f_{n-1,z_1}$$
$$g_n^{(2)} = f_{n+1,z_2} + \tfrac{1}{2}f_{n+1,z_1z_1},\qquad h_n^{(2)} = -f_{n-1,z_2} + \tfrac{1}{2}f_{n-1,z_1z_1};$$

then the t_{-2} -flow of the negative semi-discrete KP hierarchy becomes

$$D_{t_{-2}}\,\mathrm{e}^{\frac{1}{2}D_n}f_n\cdot f_n = \left(D_{z_2}+\tfrac{1}{2}D_{z_1}^2\right)\mathrm{e}^{\frac{1}{2}D_n}f_n\cdot f_n$$
$$D_yD_{z_1}f_n\cdot f_n = 2\,\mathrm{e}^{D_n}f_n\cdot f_n$$
$$D_yD_{z_2}f_n\cdot f_n = D_{z_1}\,\mathrm{e}^{-D_n}f_n\cdot f_n.$$

In general, along this line, we can construct bilinear equations with one τ -function for the negative semi-discrete KP hierarchy step by step.

Case 2.

$$\lambda = -\theta, \qquad g_n = \sum_{i=0}^{\infty} g_n^{(i)}\theta^i, \qquad h_n = \sum_{i=0}^{\infty} h_n^{(i)}\theta^i.$$

Now, we obtain another negative semi-discrete KP hierarchy

$$f_{n,t_{-m}} = f_n\Delta^{-1}\left[\frac{1}{f_{n+1}f_n}\sum_{i=0}^{m} g_n^{(i)}h_{n+1}^{(m-i)}\right],$$

$$(D_y\,\mathrm{e}^{\frac{1}{2}D_n} - \theta\,\mathrm{e}^{-\frac{1}{2}D_n} + \theta\,\mathrm{e}^{\frac{1}{2}D_n})f_n\cdot g_n = 0,$$

$$(D_y\,\mathrm{e}^{\frac{1}{2}D_n} - \theta\,\mathrm{e}^{-\frac{1}{2}D_n} + \theta\,\mathrm{e}^{\frac{1}{2}D_n})h_n\cdot f_n = 0.$$

Then, we have

$$D_{t_{-m}}f_{n+1}\cdot f_n = \sum_{i=0}^{m} g_n^{(i)}h_{n+1}^{(m-i)} \tag{29}$$

$$D_y\,\mathrm{e}^{\frac{1}{2}D_n}f_n\cdot g_n^{(i)} = (\mathrm{e}^{-\frac{1}{2}D_n} - \mathrm{e}^{\frac{1}{2}D_n})f_n\cdot g_n^{(i-1)} \tag{30}$$

$$D_y\,\mathrm{e}^{\frac{1}{2}D_n}h_n^{(i)}\cdot f_n = (\mathrm{e}^{-\frac{1}{2}D_n} - \mathrm{e}^{\frac{1}{2}D_n})h_n^{(i-1)}\cdot f_n \tag{31}$$

with $g_n^{(-1)} = h_n^{(-1)} = 0$. By the dependent variable transformation (12), we may further transform (29)–(31) into the following nonlinear form:

$$\Delta v_{n,t_{-m}} = \sum_{i=0}^{m}\phi_n^{(i)}\psi_{n+1}^{(m-i)}, \tag{32}$$

$$\phi_{n,y}^{(i+1)} = -\phi_{n+1}^{(i)} + \phi_n^{(i)} + \phi_n^{(i+1)}\Delta v_{n,y}, \tag{33}$$

$$\psi_{n+1,y}^{(i+1)} = \psi_n^{(i)} - \psi_{n+1}^{(i)} - \psi_{n+1}^{(i+1)}\Delta v_{n,y} \tag{34}$$

with $\phi_n^{(-1)} = \psi_n^{(-1)} = 0$. From (33) and (34) we see that $\phi_n^{(i+1)}$ and $\psi_n^{(i+1)}$ can be solved by y-integration in terms of $\phi_{n+1}^{(i)}, \phi_n^{(i)}$ and $\psi_{n-1}^{(i)}, \psi_n^{(i)}$, respectively. That is why we also call (32)–(34) another negative semi-discrete KP hierarchy.

Again, we construct bilinear equations with only one τ -function for the negative semi-discrete KP hierarchy (29)–(31) by introducing additional variables $z_1, z_2, \ldots$

Example 3. $m = 1$. We set $g_n^{(0)} = f_{n+1}, h_n^{(0)} = f_{n-1}, g_n^{(1)} = f_{n+1,z_1}$ and $h_n^{(1)} = -f_{n-1,z_1}$; then the t_{-1} -flow of the negative semi-discrete KP hierarchy (29)–(31) becomes

$$D_{t_{-1}}\,\mathrm{e}^{\frac{1}{2}D_n}f_n\cdot f_n = D_{z_1}\,\mathrm{e}^{\frac{1}{2}D_n}f_n\cdot f_n, \tag{35}$$

$$-D_yD_{z_1}f_n\cdot f_n = 2(\mathrm{e}^{D_n}-1)f_n\cdot f_n \tag{36}$$

which means we may choose $z_1 \equiv t_{-1}$. In this case, (35) and (36) reduce to

$$-D_yD_{t_{-1}}f_n\cdot f_n = 2(\mathrm{e}^{D_n}-1)f_n\cdot f_n \tag{37}$$

which is nothing but the bilinear form for the Toda lattice equation [19].

Example 4. $m = 2$. We set

$$g_n^{(0)} = f_{n+1}, \qquad h_n^{(0)} = f_{n-1}, \qquad g_n^{(1)} = f_{n+1,z_1}, \qquad h_n^{(1)} = -f_{n-1,z_1}$$
$$g_n^{(2)} = f_{n+1,z_2} + \tfrac{1}{2} f_{n+1,z_1z_1}, \qquad h_n^{(2)} = -f_{n-1,z_2} + \tfrac{1}{2} f_{n-1,z_1z_1};$$

then the t_{-2} -flow of the negative semi-discrete KP hierarchy (29)–(31) becomes

$$D_{t_{-2}} \mathrm{e}^{\frac{1}{2}D_n} f_n \cdot f_n = \left(D_{z_2} + \tfrac{1}{2} D_{z_1}^2\right) \mathrm{e}^{\frac{1}{2}D_n} f_n \cdot f_n$$
$$-D_y D_{z_1} f_n \cdot f_n = 2(\mathrm{e}^{D_n} - 1) f_n \cdot f_n$$
$$D_y D_{z_2} f_n \cdot f_n = D_{z_1} \mathrm{e}^{-D_n} f_n \cdot f_n.$$

Next, we show that the positive semi-discrete KP hierarchy is also derived from the same nonlocal symmetry (19) but with a different expansion.

Case 3.

$$\lambda = -\theta, \qquad g_n = \sum_{i=0}^{\infty} g_n^{(i)} \theta^{-i}, \qquad h_n = \sum_{i=0}^{\infty} h_n^{(i)} \theta^{-i}.$$

In this case, we have the following semi-discrete KP hierarchy

$$f_{n,t_m} = (-1)^m f_n \Delta^{-1} \left[\frac{1}{f_{n+1} f_n} \sum_{i=0}^{m} g_n^{(i)} h_{n+1}^{(m-i)} \right],$$
$$(D_y \mathrm{e}^{\frac{1}{2}D_n} - \theta \mathrm{e}^{-\frac{1}{2}D_n} + \theta \mathrm{e}^{\frac{1}{2}D_n}) f x_n \cdot g_n = 0,$$
$$(D_y \mathrm{e}^{\frac{1}{2}D_n} - \theta \mathrm{e}^{-\frac{1}{2}D_n} + \theta \mathrm{e}^{\frac{1}{2}D_n}) h_n \cdot f_n = 0.$$

Then, we have

$$D_{t_m} f_{n+1} \cdot f_n = (-1)^m \sum_{i=0}^{m} g_n^{(i)} h_{n+1}^{(m-i)} \tag{38}$$

$$(\mathrm{e}^{-\frac{1}{2}D_n} - \mathrm{e}^{\frac{1}{2}D_n}) f_n \cdot g_n^{(i)} = D_y \mathrm{e}^{\frac{1}{2}D_n} f_n \cdot g_n^{(i-1)} \tag{39}$$

$$(\mathrm{e}^{-\frac{1}{2}D_n} - \mathrm{e}^{\frac{1}{2}D_n}) h_n^{(i)} \cdot f_n = D_y \mathrm{e}^{\frac{1}{2}D_n} h_n^{(i-1)} \cdot f_n \tag{40}$$

with $g_n^{(-1)} = h_n^{(-1)} = 0$. By the dependent variable transformation (12), we may further transform (38)–(40) into the following nonlinear form:

$$\Delta v_{n,t_m} = (-1)^m \sum_{i=0}^{m} \phi_n^{(i)} \psi_{n+1}^{(m-i)}, \tag{41}$$

$$\phi_{n,y}^{(i)} = -\phi_{n+1}^{(i+1)} + \phi_n^{(i+1)} + \phi_n^{(i)} \Delta v_{n,y}, \tag{42}$$

$$\psi_{n+1,y}^{(i)} = \psi_n^{(i+1)} - \psi_{n+1}^{(i+1)} - \psi_{n+1}^{(i)} \Delta v_{n,y} \tag{43}$$

with $\phi_n^{(-1)} = \psi_n^{(-1)} = 0$. From (42) and (43) we see that $\phi_n^{(i+1)}$ and $\psi_n^{(i+1)}$ can be solved explicitly in terms of $\phi_n^{(i)}$ and $\psi_n^{(i)}$, respectively. That is why we call (32)–(34) a positive semi-discrete KP hierarchy.

By direct calculations, we obtain

$$g_n^{(0)} = f_n, \qquad g_n^{(1)} = f_{n,y}, \qquad g_n^{(2)} = -\tfrac{1}{2} f_{n,t} - f_{n,y} + \tfrac{1}{2} f_{n,yy}, \qquad \cdots$$

and

$$h_n^{(0)} = f_n, \qquad h_n^{(1)} = -f_{n,y}, \qquad h_n^{(2)} = \tfrac{1}{2} f_{n,t} + f_{n,y} + \tfrac{1}{2} f_{n,yy}, \qquad \cdots$$

from which we have

$$D_{t_1}\,\mathrm{e}^{\frac{1}{2}D_n} f_n\cdot f_n = D_y\,\mathrm{e}^{\frac{1}{2}D_n} f_n\cdot f_n$$

which means we may choose $t_1 \equiv y$. Besides, we have

$$\begin{cases}\left(D_{t_2}-\frac{1}{2}D_t-D_y-\frac{1}{2}D_y^2\right)\mathrm{e}^{\frac{1}{2}D_n} f_n\cdot f_n=0\\ \left(D_t+2D_y-D_y^2\right)\mathrm{e}^{\frac{1}{2}D_n} f_n\cdot f_n=0.\end{cases}\tag{44}$$

Through choosing

$$\frac{\partial}{\partial t_2}=\frac{\partial}{\partial t}+2\frac{\partial}{\partial y},$$

the system (44) reduces to the semi-discrete KP equation

$$\left(D_t+2D_y-D_y^2\right)\mathrm{e}^{\frac{1}{2}D_n} f_n\cdot f_n=0.$$

3. Nonlocal symmetries for the bilinear semi-discrete BKP equation and its application

In this section, we consider the nonlocal symmetry of the bilinear semi-discrete BKP equation and its application to deriving negative and positive semi-discrete BKP hierarchies.

The semi-discrete BKP equation (2) is transformed to

$$\left[\left(D_y-D_x^2-D_x-\frac{1}{2}\right)\mathrm{e}^{\frac{1}{2}D_n}+\frac{1}{2}\,\mathrm{e}^{\frac{3}{2}D_n}\right]f_n\cdot f_n=0\tag{45}$$

by the dependent variable transformation [18]

$$u_n=\ln\frac{f_{n+1}}{f_n}.$$

Inferring from the bilinear Bäcklund transformation of the semi-discrete BKP equation with a self-consistent source [18], we have two sets of bilinear Bäcklund transformation of equation (45) given by

$$\left(D_x+\frac{1}{2}\,\mathrm{e}^{-D_n}-\frac{1}{2}\,\mathrm{e}^{D_n}\right)g_n\cdot f_n=0\tag{46}$$

$$(2D_y-2D_x-D_x\,\mathrm{e}^{-D_n}-D_x\,\mathrm{e}^{D_n})g_n\cdot f_n=0\tag{47}$$

and

$$\left(D_x+\frac{1}{2}\,\mathrm{e}^{-D_n}-\frac{1}{2}\,\mathrm{e}^{D_n}\right)h_n\cdot f_n=0\tag{48}$$

$$(2D_y-2D_x-D_x\,\mathrm{e}^{-D_n}-D_x\,\mathrm{e}^{D_n})h_n\cdot f_n=0.\tag{49}$$

The following result holds.

Proposition 2. *If g_n and h_n satisfy (46)–(49), then σ_n satisfying*

$$\Delta(\sigma_n/f_n)=\phi_{n+1}\psi_n-\phi_n\psi_{n+1}\tag{50}$$

$$(\sigma_n/f_n)_x=\frac{1}{2}(\phi_{n+1}\psi_{n-1}-\phi_{n-1}\psi_{n+1})\,\mathrm{e}^{u_n-u_{n-1}}\tag{51}$$

$$\begin{aligned}(\sigma_n/f_n)_y=&\frac{1}{4}(\phi_{n+1}\psi_{n-2}-\phi_{n-2}\psi_{n+1}-\phi_{n+1}\psi_n+\phi_n\psi_{n+1})\,\mathrm{e}^{u_n-u_{n-2}}\\&+\frac{1}{2}(u_{n-1,x}+u_{n,x}+1)(\phi_{n+1}\psi_{n-1}-\phi_{n-1}\psi_{n+1})\,\mathrm{e}^{u_n-u_{n-1}}\\&+\frac{1}{4}(\phi_{n+2}\psi_{n-1}-\phi_{n-1}\psi_{n+2}-\phi_n\phi_{n-1}+\phi_{n-1}\psi_n)\,\mathrm{e}^{u_{n+1}-u_{n-1}}\end{aligned}\tag{52}$$

is a non-local symmetry of the bilinear semi-discrete BKP equation (45), where

$$u_n = \Delta v_n = \ln \frac{f_{n+1}}{f_n}, \qquad \phi_n = \frac{g_n}{f_n}, \qquad \psi_n = \frac{h_n}{f_n}. \tag{53}$$

This means that σ_n satisfies the symmetry equation

$$\left[\left(D_y - D_x^2 - D_x - \tfrac{1}{2}\right) \mathrm{e}^{\frac{1}{2}D_n} + \tfrac{1}{2}\,\mathrm{e}^{\frac{3}{2}D_n}\right](f_n \cdot \sigma_n + \sigma_n \cdot f_n) = 0. \tag{54}$$

Proof. Under the dependent variable transformations (53), the bilinear semi-discrete BKP equation (45) is transformed to

$$\Delta v_{n,y} - (\Delta v_{n,x})^2 - (2+\Delta) v_{n,xx} - \Delta v_{n,x} - \tfrac{1}{2} + \tfrac{1}{2}\,\mathrm{e}^{u_{n+1}-u_{n-1}} = 0, \tag{55}$$

while two bilinear Bäcklund transformations (46)–(49) are transformed to, respectively,

$$\phi_{n,x} = \tfrac{1}{2}(\phi_{n+1} - \phi_{n-1})\,\mathrm{e}^{u_n - u_{n-1}} \tag{56}$$

$$\phi_{n,y} = \phi_{n,x} + \tfrac{1}{2}[\phi_{n-1,x} + \phi_{n+1,x} + (\phi_{n+1} - \phi_{n-1})(u_{n,x} + u_{n-1,x})]\,\mathrm{e}^{u_n - u_{n-1}} \tag{57}$$

and

$$\psi_{n,x} = \tfrac{1}{2}(\psi_{n+1} - \psi_{n-1})\,\mathrm{e}^{u_n - u_{n-1}} \tag{58}$$

$$\psi_{n,y} = \psi_{n,x} + \tfrac{1}{2}[\psi_{n-1,x} + \psi_{n+1,x} + (\psi_{n+1} - \psi_{n-1})(u_{n,x} + u_{n-1,x})]\,\mathrm{e}^{u_n - u_{n-1}}. \tag{59}$$

All the above equations guarantee that (50)–(52) are compatible. Furthermore, let $w_n = \sigma_n / f_n$ for convenience, then we have

$$\begin{aligned}
&\frac{1}{f_{n+1} f_n}\left[\left(D_y - D_x^2 - D_x - \frac{1}{2}\right) \mathrm{e}^{\frac{1}{2}D_n} + \frac{1}{2}\,\mathrm{e}^{\frac{3}{2}D_n}\right](f_n \cdot \sigma_n + \sigma_n \cdot f_n) \\
&\quad = \Delta w_{n,y} - (2+\Delta) w_{n,xx} - 2(\Delta w_{n,x})\Delta v_{n,x} - \Delta w_{n,x} + \frac{1}{2}(\Delta w_{n+1} \\
&\qquad - \Delta w_{n-1})\,\mathrm{e}^{u_{n+1}-u_{n-1}} + (w_{n+1} + w_n)[\Delta v_{n,y} - (\Delta v_{n,x})^2 - (2+\Delta) v_{n,xx} \\
&\qquad - \Delta v_{n,x} - \frac{1}{2} + \frac{1}{2}\,\mathrm{e}^{u_{n+1}-u_{n-1}}] \\
&\quad = \Delta w_{n,y} - (2+\Delta) w_{n,xx} - 2(\Delta w_{n,x})\Delta v_{n,x} - \Delta w_{n,x} \\
&\qquad + \frac{1}{2}(\Delta w_{n+1} - \Delta w_{n-1})\,\mathrm{e}^{u_{n+1}-u_{n-1}}.
\end{aligned}$$

Substituting (50)–(52) into the above expression, we conclude that the symmetry equation holds. □

For convenience, we denote σ_n as

$$\sigma_n = f_n \Delta^{-1} \frac{g_{n+1} h_n - g_n h_{n+1}}{f_{n+1} f_n}. \tag{60}$$

Since the bilinear BKP equation (45) is invariant under the gauge transformation

$$f_n \to \epsilon^n \mathrm{e}^{\lambda x + \theta y} f_n,$$

three arbitrary parameters can be introduced in the nonlocal symmetry by

$$g_n \to \epsilon^n \mathrm{e}^{\lambda x + \theta y} g_n, \qquad h_n = \epsilon^{-n}\,\mathrm{e}^{-\lambda x - \theta y}.$$

we have from (60) and (46)–(49)

$$\sigma_n = f_n \Delta^{-1} \frac{\epsilon g_{n+1} h_n - \frac{1}{\epsilon} g_n h_{n+1}}{f_{n+1} f_n} \tag{61}$$

is nonlocal symmetries of the bilinear semi-discrete BKP equation (45), where g_n, h_n satisfy, respectively,

$$\left(D_x + \frac{1}{2\epsilon}\,\mathrm{e}^{-D_n} - \frac{\epsilon}{2}\,\mathrm{e}^{D_n} + \lambda\right) g_n \cdot f_n = 0 \tag{62}$$

$$\begin{aligned}&(2D_y - 2D_x - \frac{1}{\epsilon}D_x\,\mathrm{e}^{-D_n} - \epsilon D_x\,\mathrm{e}^{D_n} - \frac{\lambda}{\epsilon}\,\mathrm{e}^{-D_n}\\&-\epsilon\lambda\,\mathrm{e}^{D_n} + 2\theta - 2\lambda)g_n \cdot f_n = 0\end{aligned} \tag{63}$$

and

$$\left(D_x + \frac{\epsilon}{2}\,\mathrm{e}^{-D_n} - \frac{1}{2\epsilon}\,\mathrm{e}^{D_n} - \lambda\right) h_n \cdot f_n = 0 \tag{64}$$

$$\begin{aligned}&(2D_y - 2D_x - \epsilon D_x\,\mathrm{e}^{-D_n} - \frac{1}{\epsilon}D_x\,\mathrm{e}^{D_n} + \epsilon\lambda\,\mathrm{e}^{-D_n}\\&+\frac{\lambda}{\epsilon}\,\mathrm{e}^{D_n} - 2\theta + 2\lambda)h_n \cdot f_n = 0.\end{aligned} \tag{65}$$

In the following, we would like to derive a hierarchy of negative semi-discrete BKP equations.

Case 1.

$$\epsilon = 1, \qquad g_n = \sum_{i=0}^{\infty} g_n^{(i)}\lambda^i, \qquad h_n = \sum^{\infty} h_n^{(i)}\lambda^i.$$

In this case, we have the negative semi-discrete BKP hierarchy

$$\begin{aligned}&f_{n,t_{-m}} = f_n\Delta^{-1}\left\{\frac{1}{f_{n+1}f_n}\sum_{i=0}^{m}\left[g_{n+1}^{(i)}h_n^{(m-i)} - g_n^{(i)}h_{n+1}^{(m-i)}\right]\right\}\\&\left(D_x + \frac{1}{2}\,\mathrm{e}^{-D_n} - \frac{1}{2}\,\mathrm{e}^{D_n} + \lambda\right) g_n \cdot f_n = 0\\&\left(D_x + \frac{1}{2}\,\mathrm{e}^{-D_n} - \frac{1}{2}\,\mathrm{e}^{D_n} - \lambda\right) h_n \cdot f_n = 0.\end{aligned}$$

Then, we have

$$D_{t_{-m}} f_{n+1} \cdot f_n = \sum_{i=0}^{m}\left[g_{n+1}^{(i)}h_n^{(m-i)} - g_n^{(i)}h_{n+1}^{(m-i)}\right] \tag{66}$$

$$\left(D_x + \frac{1}{2}\,\mathrm{e}^{-D_n} - \frac{1}{2}\,\mathrm{e}^{D_n}\right) g_n^{(i)} \cdot f_n = -g_n^{(i-1)} \cdot f_n \tag{67}$$

$$\left(D_x + \frac{1}{2}\,\mathrm{e}^{-D_n} - \frac{1}{2}\,\mathrm{e}^{D_n}\right) h_n^{(i)} \cdot f_n = h_n^{(i-1)} \cdot f_n \tag{68}$$

with $g_n^{(-1)} = 0$, $h_n^{(-1)} = 0$.

Again, we show how to rewrite (66)–(68) into bilinear equations with only one τ -function through examples.

Example 5. $m = 1$. In this case, let

$$g_n^{(0)} = f_n, \qquad h_n^{(0)} = f_n, \qquad g_n^{(1)} = f_{n,z_1}, \qquad h_n^{(1)} = -f_{n,z_1}. \tag{69}$$

The t_{-1} -flow of negative semi-discrete BKP hierarchy (66)–(68) becomes

$$D_{t_{-1}}\,\mathrm{e}^{\frac{1}{2}D_n} f_n \cdot f_n = 2D_{z_1}\,\mathrm{e}^{\frac{1}{2}D_n} f_n \cdot f_n$$
$$D_x D_{z_1} f_n \cdot f_n + D_{z_1}\,\mathrm{e}^{-D_n} f_n \cdot f_n = -2 f_n \cdot f_n,$$

i.e.,

$$D_x D_{t_{-1}} f_n \cdot f_n + D_{t_{-1}}\,\mathrm{e}^{-D_n} f_n \cdot f_n = -4 f_n \cdot f_n. \tag{70}$$

In the same way, the negative semi-discrete BKP hierarchy (66)–(68) can be rewritten in terms of only one τ -function step by step.

Finally, we want to mention that the semi-discrete BKP hierarchy can be derived from the nonlocal symmetry (61).

Case 2.

$$\lambda = \frac{\epsilon}{2} - \frac{1}{2\epsilon}, \qquad g_n = \sum^{\infty} g_n^{(i)} \epsilon^i, \qquad h_n = \sum^{\infty} h_n^{(i)} \epsilon^i.$$

We have the following semi-discrete BKP hierarchy

$$f_{n,t_m} = -\frac{1}{2} f_n \Delta^{-1} \left[\frac{1}{f_{n+1} f_n} \left(\sum_{i=0}^{m-2} g_{n+1}^{(i)} h_n^{(m-2-i)} + \sum_{i=0}^{m} g_n^{(i)} h_n^{(m-i)} \right) \right] \tag{71}$$

$$\left(D_x + \frac{1}{2\epsilon} (\mathrm{e}^{-D_n} - 1) - \frac{\epsilon}{2} (\mathrm{e}^{D_n} - 1) \right) g_n \cdot f_n = 0 \tag{72}$$

$$\left(D_x + \frac{\epsilon}{2} (\mathrm{e}^{-D_n} - 1) - \frac{1}{2\epsilon} (\mathrm{e}^{D_n} - 1) \right) h_n \cdot f_n = 0. \tag{73}$$

Then, we have

$$D_{t_m}\,\mathrm{e}^{\frac{1}{2}D_n} f_{n+1} \cdot f_n = -\frac{1}{2} \left(\sum_{i=0}^{m-2} g_{n+1}^{(i)} h_n^{(m-2-i)} + \sum_{i=0}^{m} g_n^{(i)} h_n^{(m-i)} \right) \tag{74}$$

$$(\mathrm{e}^{-D_n} - 1) g_n^{(i)} \cdot f_n = (\mathrm{e}^{D_n} - 1) g_n^{(i-2)} \cdot f_n - 2D_x g_n^{(i-1)} \cdot f_n \tag{75}$$

$$(\mathrm{e}^{D_n} - 1) h_n^{(i)} \cdot f_n = (\mathrm{e}^{-D_n} - 1) h_n^{(i-2)} \cdot f_n + 2D_x h_n^{(i-1)} \cdot f_n \tag{76}$$

with $g_n^{(-2)} = g_n^{(-1)} = 0$, $h_n^{(-2)} = h_n^{(-1)} = 0$.

By direct calculations, we have

$g_n^{(0)} = f_{n+1}$, $\quad g_n^{(1)} = 2f_{n+1,x}$, $\quad g_n^{(2)} = 2f_{n+1,xx} + 2f_{n+1,y} - 2f_{n+1,x} - \frac{1}{2} f_{n+1}$

$h_n^{(0)} = f_{n-1}$, $\quad g_n^{(1)} = -2f_{n-1,x}$, $\quad g_n^{(2)} = 2f_{n-1,xx} - 2f_{n-1,y} + 2f_{n-1,x} - \frac{1}{2} f_{n-1}$

and

$$D_{t_1}\,\mathrm{e}^{\frac{1}{2}D_n} f_n \cdot f_n = D_x\,\mathrm{e}^{\frac{1}{2}D_n} f_n \cdot f_n \tag{77}$$

which means we may choose $t_1 \equiv x$. Besides, we have

$$\begin{cases} \left[\left(D_{t_2} - D_y - D_x^2 + D_x - \frac{1}{2} \right) \mathrm{e}^{\frac{1}{2}D_n} + \frac{1}{2}\,\mathrm{e}^{\frac{3}{2}D_n} \right] f_n \cdot f_n = 0 \\ \left[\left(D_y - D_x^2 - D_x - \frac{1}{2} \right) \mathrm{e}^{\frac{1}{2}D_n} + \frac{1}{2}\,\mathrm{e}^{\frac{3}{2}D_n} \right] f_n \cdot f_n = 0. \end{cases} \tag{78}$$

By choosing

$$\frac{\partial}{\partial t_2} = 2\frac{\partial}{\partial y} - 2\frac{\partial}{\partial x},$$

the system (78) reduces to the semi-discrete BKP equation

$$\left[\left(D_y - D_x^2 - D_x - \frac{1}{2} \right) \mathrm{e}^{\frac{1}{2}D_n} + \frac{1}{2}\,\mathrm{e}^{\frac{3}{2}D_n} \right] f_n \cdot f_n = 0.$$

4. Conclusion and discussions

In this paper, we have investigated nonlocal symmetries for the bilinear semi-discrete KP and bilinear semi-discrete BKP equations. By expanding these nonlocal symmetries, we have derived bilinear negative semi-discrete KP and BKP hierarchies. The interesting thing is that bilinear positive and negative semi-discrete KP hierarchies may be derived from the same nonlocal symmetries for the semi-discrete KP equation. It is remarked that the first members of the obtained negative semi-discrete KP hierarchies correspond to the Toda molecule equation and the Toda lattice equation, respectively. So these two negative semi-discrete KP hierarchies are the same in nonlinear form (up to simple variable transformation). Besides, we have written nonlinear forms for the negative and positive semi-discrete KP hierarchies. Similarly we may also derive nonlinear forms for the negative and positive semi-discrete BKP hierarchies. Based on the fact that recursion operators have played an important role in expressing positive and negative (1+1)-dimensional integrable hierarchies of equations, it is natural to inquire if we can also express negative and positive semi-discrete KP (and semi-discrete BKP) hierarchies in terms of its corresponding recursion operator. Unfortunately, until now we have not known the explicit form of the recursion operator for the semi-discrete KP equation. We hope that a recursion operator for the semi-discrete KP hierarchy can be worked out later on. Finally it is noted that some important work has been done on recursion operators in a (2+1)-dimensional continuous case [20, 21].

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